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Limit distribution of an infinite-range random Ising model

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Abstract. The limit distribution of a random Ising model (site disorder) with an exchange interaction of infinite range is calculated exactly. It is shown that randomness changes the analytic properties of the limit distribution at the critical point considerably. At criticality this system cannot be described by a Hamiltonian which is a polynomial in the spin variables.

1. Introduction

The renormalisation group theory of Kadanoff, Wilson and Fisher has been very successful in describing critical properties of homogeneous ferromagnetic systems. In contrast to this, critical properties of inhomogeneous ferromagnetic systems, specifically those with frozen disorder, are less well understood. Early attempts (for instance Grinstein and Luther 1975) fail to describe the experiments properly, later attempts (Sobotta and Wagner 1978), although in better agreement, exhibit divergencies which are difficult to understand and are difficult to remove. All these attempts to describe the critical properties of random spin systems are based on a Ginzburg–Landau–Wilson type of Hamiltonian, i.e. on a Hamiltonian which is essentially a polynomial in the spin variable.

In this rather confusing situation one might be inclined to question the validity of this kind of approach and look for a different one which is less intuitive and more rigorous. Therefore we use the probabilistic approach to critical phenomena (for a review compare Cassandro and Jona-Lasinio 1978), which deals with the distribution function of strongly dependent random variables—for instance spins of a ferromagnetically coupled Ising system—in the thermodynamic limit. Generally, it turns out that for temperatures T, which differ from the critical temperature T_c, the limit distribution is a Gaussian one, which is well known from weakly dependent or independent random variables (central limit theorem, Renyi 1973). At $T = T_c$, however, one finds a non-Gaussian distribution, in general. Of course, the two approaches are connected (Sinai 1978) and realistic physical problems are not easier to handle in the probabilistic approach than in the other one.

However, it has been shown recently (Ellis and Newman 1978, Ellis *et al* 1980) that even in the unphysical limit of infinite range of interaction (mean field approximation) the limit distribution of an Ising ferromagnet can be non-Gaussian at $T = T_c$, if the limit $N \to \infty$ is taken appropriately. It may therefore be interesting to study the influence of randomness on the distribution function in this non-trivial limit.

2. The infinite-range Ising model

We deal with a ferromagnetic Ising model with site disorder. Each site i of a lattice can be occupied by a spin S_i with probability p. If K_i is a random occupation number, which takes the value $K_i = 0$ if i is unoccupied, and $K_i = 1$ if i is occupied by a spin, the Hamiltonian of the system is given by

$$\mathscr{H} = -\tfrac{1}{2}\sum_{i,j=1}^{N} J_{ij}K_iS_iK_jS_j - H\sum_{i=1}^{N} K_iS_i \tag{2.1}$$

with $S_i = \pm 1, J_{ij} > 0$. H is the external magnetic field and N is the number of sites of the lattice.

The probability P that the system is in a definite configuration of spins, $\{S_i\}$, and of occupation numbers, $\{K_i\}$, (= distribution of the spins over the lattice) is given by

$$P(\{K_i\}, \{S_i\}) = p(\{K_i\})P(\{S_i\}|\{K_i\}) \tag{2.2}$$

where $P(\{S_i\}|\{K_i\})$ is the conditional probability that for a given spatial distribution of spins over the lattice, $\{K_i\}$, a definite spin configuration $\{S_i\}$ can be found. Obviously, the conditional probability is given by the usual Boltzmann factor

$$P(\{S_i\}|\{K_i\}) = Z_c^{-1}\exp(-\beta\mathscr{H}_c) \tag{2.3}$$

where c is a definite spatial configuration, $c = \{K_i\}$, and where H_c is the appropriate Hamiltonian (energy) and $Z_c = \mathrm{Tr}\exp(-\beta\mathscr{H}_c)$ is the related partition function. β is the inverse temperature. The *a priori* probability $p(\{K_i\})$ for a definite distribution of randomly distributed spins over the lattice is given by

$$p(\{K_i\}) = \prod_i (p\delta_{K_i,1} + (1-p)\delta_{K_i,0}). \tag{2.4}$$

In the limit of an exchange interaction J_{ij} with infinite range we put, to ensure extensivity of the energy, $J_{ij} = (1/N)J$. Instead of equation (2.1) we then have for the Hamiltonian

$$\mathscr{H} = -\frac{J}{2N}\left(\sum_{i=1}^{N} K_iS_i\right)^2 - H\sum_{i=1}^{N} K_iS_i = -\frac{J}{2N}\left(\sum_{i=1}^{n} S_i\right)^2 - H\sum_{i=1}^{n} S_i \tag{2.5}$$

where n is the number of occupied sites.

Now we have essentially two different random variables; accordingly we define as random variables

$$X = \frac{1}{N^{\kappa/2}}\sum_{i=1}^{N}(K_i - \langle K_i\rangle) = \frac{1}{N^{\kappa/2}}\sum_{i=1}^{N}(K_i - p) \tag{2.6}$$

where $\langle K_i\rangle = p$ is the configuration averaged mean value of the occupation number and where κ is the critical index associated with X. Obviously, X can take values x

$$x = N^{-\kappa/2}(n - pN) \tag{2.7}$$

with $0 \leqslant n \leqslant N$, where n is an integer.

The second random variable refers to the spins and is defined by

$$Y = \frac{1}{N^{\rho/2}}\sum_{i=1}^{N}(K_iS_i - \langle K_iS_i\rangle) = \frac{1}{N^{\rho/2}}\left(\sum_{i=1}^{n} S_i - N\sigma_0\right) \tag{2.8}$$

where σ_0 is the mean value of K_iS_i, i.e. the mean value of the magnetisation per site.

ρ is the critical index associated with Y. The values of κ and ρ are in the interval (1, 2) (Cassandro and Jona-Lasinio 1978). If m is the number of down spins and $n-m$ the number of up spins for a given distribution of spins, characterised by n, Y can take values y given by

$$y = N^{-\rho/2}(n-2m-N\sigma_0) \tag{2.9}$$

with $0 \leqslant m \leqslant n$.

Now, the probability that Y takes a value y and X a value x is given, according to equations (2.1)–(2.3), by

$$P(x,y) = p(x)P(y|x) = p(x)\frac{\binom{n}{m}\exp\left(\frac{\beta}{2N}(n-2m)^2+\beta H(n-2m)\right)}{\sum_{m=0}^{n}\binom{n}{m}\exp\left(\frac{\beta}{2N}(n-m)^2+\beta H(n-2m)\right)} \tag{2.10}$$

where $p(x)$ is the probability that n sites are occupied by spins in a lattice of N sites,

$$p(x) = \binom{N}{n}p^n(1-p)^{N-n} \tag{2.11}$$

with x given by equation (2.7). In equation (2.10) the exchange constant J has been absorbed into β and H respectively. The denominator of equation (2.10) corresponds to the partition function $Z(x)$ of a lattice with n spins.

3. The limit distribution

Since the conditional probability $P(y|x)$ is normalised to 1, $\Sigma_m P(y|x) = 1$, the limit distribution $p(x)$ as given by equation (2.11) is the Gaussian distribution of independent random variables (for instance, Renyi 1973), given by

$$p(x) = [2\pi Np(1-p)]^{-1/2}\exp[-x^2/2p(1-p)] \tag{3.1}$$

for $N \to \infty$ and with $\kappa = 1$.

Similarly, utilising Stirling's formula, we find after some straightforward calculation

$$\begin{aligned}P(y|x) = Z^{-1}(x)\exp\Big[&N^{\kappa/2}x[\ln Np - \tfrac{1}{2}\ln\tfrac{1}{4}N^2(p^2-\sigma_0^2)]\\
&-\frac{N^{\kappa-1}}{2}x^2\frac{\sigma_0^2}{p(p^2-\sigma_0^2)} + N^{\rho/2}y\left(-\frac{1}{2}\ln\frac{p+\sigma_0}{p-\sigma_0}+\beta\sigma_0+\beta H\right)\\
&+\tfrac{1}{2}N^{\rho-1}y^2\left(\beta-\frac{p}{p^2-\sigma_0^2}\right) + N^{(\kappa+\rho)/2-1}\frac{\sigma_0 xy}{p^2-\sigma_0^2}\\
&+\frac{N^{\kappa+\rho/2-2}}{4}x^2y\left(\frac{1}{(p+\sigma_0)^2}-\frac{1}{(p-\sigma_0)^2}\right)\\
&+\frac{N^{\rho+\kappa/2-2}}{4}xy^2\left(\frac{1}{(p+\sigma_0)^2}+\frac{1}{(p-\sigma_0)^2}\right),\end{aligned}$$

$$+\frac{N^{3\rho/2-2}}{12}y^3\left(\frac{1}{(p+\sigma_0)^2}-\frac{1}{(p-\sigma_0)^2}\right)-\frac{N^{2\rho-3}}{24}y^4\left(\frac{1}{(p+\sigma_0)^3}+\frac{1}{(p-\sigma_0)^3}\right)$$
$$-\frac{N^{3\rho/2+\kappa/2-3}}{6}xy^3\left(\frac{1}{(p+\sigma_0)^3}-\frac{1}{(p-\sigma_0)^3}\right)$$
$$-\frac{N^{\kappa+\rho-3}}{4}x^2y^2\left(\frac{1}{(p+\sigma_0)^3}+\frac{1}{(p-\sigma_0)^3}\right)\Bigg] \tag{3.2}$$

with $\kappa=1$. Terms which vanish in the limit $N\to\infty$ for $\rho\leq 2$ have been left out. In order to get a normalisable distribution function for $N\to\infty$ the factor of $N^{\rho/2}y$ in the exponent has to vanish, which gives

$$\frac{1}{2}\ln\frac{p+\sigma_0}{p-\sigma_0}-\beta\sigma_0=\beta H. \tag{3.3}$$

This gives the magnetisation σ_0 as a function of T and H. For $p=1$ equation (3.3) is the well known mean field result for σ_0. For $p<1$ one can put $\sigma_0=p\sigma$, where σ is now given by the mean field equation of the homogeneous system with $p=1$ but with an exchange constant rescaled by a factor p.

For $\sigma_0>0$ and $H\geq 0$ one has from equation (3.3)

$$\beta\sigma_0\leq\frac{1}{2}\ln\frac{p+\sigma_0}{p-\sigma_0}<\frac{p\sigma_0}{p^2-\sigma_0^2} \qquad \text{or} \qquad \beta<\frac{p}{p^2-\sigma_0^2}.$$

Therefore in the region $\sigma_0>0$, which covers the ferromagnetic region as well as the paramagnetic region, not including the critical point, a stable limit distribution requires $\rho=1$, as is obvious from equation (3.2). Accordingly one gets a Gaussian limit distribution

$$P(x,y)=\frac{1}{\pi}\left(\frac{p-\beta(p^2-\sigma_0^2)}{N^2(p^2-\sigma_0^2)p(1-p)}\right)^{1/2}\exp\Bigg[-\frac{x^2}{2p(1-p)}$$
$$-\frac{1}{2(p^2-\sigma_0^2)}\left(\frac{\sigma_0 x}{[p-\beta(p^2-\sigma_0^2)]^{1/2}}-y[p-\beta(p^2-\sigma_0^2)]^{1/2}\right)^2\Bigg]. \tag{3.4}$$

Integrating over x gives the effective limit distribution for y,

$$P(y)=\sqrt{N}\int_{-\infty}^{+\infty}\mathrm{d}x\,P(x,y),$$

$$P(y)=\left(\frac{2}{\pi N}\right)^{1/2}\frac{p-\beta(p^2-\sigma_0^2)}{\{(p^2-\sigma_0^2)[p-\beta(p^2-\sigma_0^2)+\sigma_0^2p(1-p)]\}^{1/2}}$$
$$\times\exp\left(-\frac{y^2}{2}\frac{[p-\beta(p^2-\sigma_0^2)]^2}{(p^2-\sigma_0^2)[p-\beta(p^2-\sigma_0^2)]+\sigma_0^2p(1-p)}\right). \tag{3.5}$$

From this one derives the fluctuation of y

$$\langle y^2\rangle=\frac{(p^2-\sigma_0^2)[p-\beta(p^2-\sigma_0^2)]+\sigma_0^2p(1-p)}{[p-\beta(p^2-\sigma_0^2)]^2}$$

which reduces in the paramagnetic region ($H=0$) to

$$\langle y^2\rangle = p/(1-\beta p). \tag{3.6}$$

Obviously, randomness reduces the fluctuations of the order parameter.

At the critical point $H=0$, $\sigma_0=0$, $\beta p=1$ a different, non-Gaussian limit distribution emerges from equation (3.2) with $\rho=\frac{3}{2}$ which is given by

$$P(x\,|\,y) = C(x)\exp(xy^2/2p^2 - y^4/12p^3) \tag{3.7}$$

with

$$C^{-1}(x) = \frac{[12(Np)^3]^{1/4}}{2}\int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}}\exp\left[-t^2+\left(\frac{3}{p}\right)^{1/2}xt\right]. \tag{3.8}$$

$C^{-1}(x)$ is an entire function of x which can be expressed in terms of the parabolic cylinder function $D_{-1/2}$. Doing this, one finds

$$P(x,y) = \frac{2}{\pi[2Np(1-p)]^{1/2}(6N^3p^3)^{1/4}} \times \exp\left\{-\frac{x^2}{2p(1-p)} - \frac{3x^2}{8p} + \frac{xy^2}{2p^2} - \frac{y^4}{12p^3} - \ln D_{-1/2}\left[-\left(\frac{3}{2p}\right)^{1/2}x\right]\right\}.$$

Obviously, $\ln P(x,y)$ is not a polynomial in x and y.

Integrating over x one gets the effective limit distribution $P(y)$:

$$P(y) = N^{1/2}\int_{-\infty}^{+\infty}\mathrm{d}x\,P(x,y)$$

$$= \frac{2}{\pi^{1/2}(12p^3N^3)^{1/4}}\exp\left(-\frac{y^4}{12p^3}\right)\int_{-\infty}^{+\infty}\frac{\exp[-x^2+\alpha xy^2/(12p^3)^{1/2}]}{\int_0^\infty \mathrm{d}t\,t^{-1/2}\exp(-t^2+\alpha xt)}\,\mathrm{d}x \tag{3.10}$$

with $\alpha=[6(1-p)]^{1/2}$. The integral contains, apart from the p dependence of the prefactors, the corrections to the limit distribution caused by the randomness of the system. Obviously, $\ln P(y)$ is not a polynomial in y. For $y\to\infty$ the integrals can be performed asymptotically. Utilising Laplace's method to evaluate the integrals, one finds

$$\lim_{y\to\infty} P(y) \sim |y|\exp[-y^4/6p^3(5-3p)]. \tag{3.11}$$

The fluctuation of y cannot be calculated from equation (3.10) exactly. For $\alpha \ll 1$ or $p\cong 1$, however, one finds

$$\langle y^2\rangle = (6^{1/2}/\pi)\Gamma(\tfrac{3}{4})[1+\tfrac{3}{2}(1-p)(\Gamma^4(\tfrac{3}{4})/\pi^2-1)] \tag{3.12}$$

with $\Gamma(\frac{3}{4})=1.225\ldots$ (Abramowitz and Stegun 1965). Again, $\langle y^2\rangle$ is a decreasing function of p.

4. Results and conclusions

The effect of the random distribution of spins in this model is quite clear: the critical indices are not changed when going from a homogeneous system to a non-homogeneous, random system. However, the limit distributions are quite different.

For $T \neq T_c$ the limit distribution remains a Gaussian one, as expected, and only details are different. Fluctuations of the order parameter are smaller than in the homogeneous system.

At the critical point, the effect of randomness on the limit distribution is much more pronounced. In the homogeneous case, $\ln P(y)$ is a polynomial in y of order four; in the inhomogeneous case the analytic properties of $\ln P(y)$ are completely changed since it can no longer be represented by a polynomial.

Although the calculations cannot be extended to the more relevant and more complicated case of short-range interaction, it seems to be rather obvious that the main result of this calculation will be preserved in systems with short-range interaction, namely that randomness will cause a drastic change of the analytic properties of the limit distribution at the critical point. If this conjecture is correct, it is quite clear that any theory based on a simple Ginzburg–Landau–Wilson Hamiltonian for the effective partition function of a random spin system (for instance Grinstein and Luther 1975) cannot be equivalent to a limit distribution $P(y)$ of the kind found in this paper and cannot therefore be correct. Obviously, one has to take into account an infinite number of products of spin operators, i.e. the Hamiltonian near criticality is probably a transcendental function of the spin operators.

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